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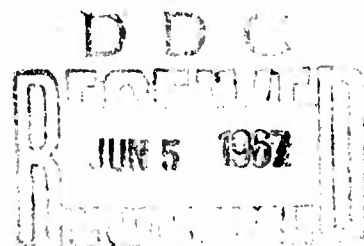## SCIENTIFIC RESEARCH LABORATORIES

A Probabilistic Interpretation of Miner's Rule

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32

A PROBABILISTIC INTERPRETATION OF MINER'S RULE

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Summary

Miner's rule for the cumulative damage due to fatigue, the behavior of which is well known in engineering practice as a deterministic rule, is examined from a probabilistic point of view. By adopting a model for stochastic crack growth with incremental extensions having a distribution with increasing failure rate, and utilizing some results from renewal theory, we exhibit conditions of dependence upon load under which Miner's rule does yield the mathematical expectation of the fatigue life. We also obtain conditions of dependence under which it is conservative and others when it is unconservative. The relationships between the mathematical assumptions which govern when the rule is, *on the average,* conservative or unconservative, are related to the physical conditions in practice which are known to force significant departures from the rule.

1

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1. Introduction

In one of the first papers dealing with the prediction of fatigue life, M. A. Miner [6] proposed over 21 years ago a criterion which has become famous (or infamous) as "Miner's rule". Dealing with fatigue as if it were a deterministic phenomenon and using such non-quantifiable concepts as "internal work" and "damage" the author derived heuristically a rule which has been applied ɔ the prediction of fatigue life under repeated cyclic loadings with varying stresses.

It was soon experimentally verified that Miner's rule could not give a fatigue life prediction under all types of loading, which would be sufficiently accurate to be used without modification as an engineering design tool. Thus, since its formulation in 1945 there have been a score or so other deterministic rules proposed to take its place. However, because Miner's rule was so simple to apply and seemed to give no worse results than any of the others, it has remained as a standard for comparison. In fact, calculations performed on some extensive empirical data [3] showed that in many practical cases the rule yields approximately correct and rather conservative results, i.e. that it tends to underestimate the true number of cycles to failure. On the other hand, recent experimental studies carried out under carefully controlled conditions, see [4], [5] and the references given there, have resulted in data which make it seem likely that the applicability of Miner's rule depends on the arrangement of the loading sequence within the cycle, and that under some circumstances this rule may substantially overestimate the true number of cycles to failure.

2. The Traditional Derivation of Miner's Rule

We consider standardized specimens of a material which are exposed
to fluctuating stresses due to periodic loading. To be specific by a *load*
(or *load oscillation*) we mean a continuous unimodal function on the unit
interval, the value of which at any time gives the stress imposed by
deflection of the material specimen. Thus our load function contains all
the information needed to determine such parameters as maximum stress,
minimum stress and average stress which are usually used to define the
loading oscillation.

The assumptions usually made to obtain Miner's rule may be
summarized as follows:

(a) The amount of damage absorbed by the material in any
one oscillation is determined only by the load during
that oscillation.

(b) Each specimen can absorb the same amount of damage and
when that amount is attained, failure occurs.
This amount will be called the total damage at failure.

(c) The total damage absorbed by the specimen under a
sequence of load oscillations is equal to the sum of
damages absorbed in each oscillation during the sequence.

We introduce the following notation:

W = total damage at failure

w_i = amount of damage during one repetition of load oscillation ℓ_i

N_i = number of oscillations to failure under repeated application
of the same load ℓ_i

where there are only $i=1,\ldots,k$ possible loads considered.

Under (a), (b), (c) there follows immediately the equality

$$(2.1) \qquad W = N_i w_i \quad \text{for each} \quad i=1,\ldots,k.$$

If a sequence of loads is applied which are different from oscillation to oscillation, and failure occurs after load ℓ_1 was applied in n_1 oscillations and ℓ_2 in n_2, etc., we must have

$$(2.2) \qquad \sum_{1}^{k} n_i w_i = W$$

and from (2.1) and (2.2) follows

$$(2.3) \qquad \sum_{i=1}^{k} \frac{n_i}{N_i} = 1$$

as a necessary and sufficient condition for failure to occur after the number of oscillations to failure given by

$$(2.4) \qquad N = \sum_{i=1}^{k} n_i.$$

The most frequently used form of Miner's Rule is the following: If a given cycle contains various numbers of different oscillations say n_i oscillations of load ℓ_i for $i=1,\ldots,k$ then the number of such cycles that can be repeated until failure is

$$(2.5) \qquad N = \frac{1}{\displaystyle\sum_{i=1}^{k} \frac{n_i}{N_i}}.$$

Hence by the cumulative damage rule we determine the fraction of damage accrued during one cycle and use its reciprocal to estimate the total life. In practice N_i are determined from available S-N data and

the n_i are calculated from a typical spectrum of loading during the cycle and then we use (2.5) to estimate N.

The assumptions underlying this traditional derivation of Miner's rule are strictly speaking not verifiable and the conclusion derived, namely the deterministic formulation of the rule, is acceptable only as a first approximation. The following comments about assumptions (a), (b) and (c) make this point more specific.

Firstly the vague concept of "damage" is in need of re-interpretation in the light of modern fractography. Secondly, instead of assuming, as in (a) that constant repetition of the same load ℓ_i on each oscillation should contribute exactly the same amount w_i of "damage", it would appear more plausible to make the assumption that the "damage" caused in any single oscillation might vary from one oscillation to the other and depend upon factors other than the load at that oscillation. Similarly instead of (b) one would prefer to assume that the quantity W, the total "damage" at failure might also be a random variable which can assume different values for different specimens of the material. Finally, assumption (c) that the "damage" accruing in the jth oscillation is linearly additive to the "damages" sustained in the preceding oscillations appears also to be in need of re-interpretation and it would be desirable to modify it.

In the next section these assumptions are modified in the sense suggested above and some mathematical results are obtained which are analogous to Miner's rule in its traditional form. But these are stated in terms of mean values of the random numbers of cycles to failure and are derived under conditions which appear reasonable, at least when failure is a state attained during an early stage of the fatigue process.

3. A Probabilistic Model I

We assume that fatigue failure is due to the growth and ultimate extension of a dominant crack. At each oscillation of the imposed stress this crack is extended by some amount which is a random function, due to the variation in the material and to the influence of environment, of the magnitude of the imposed stress as well as the geometry of the specimen. The extension of the crack at each oscillation is therefore a non-negative random variable whose distribution may depend upon several parameters the nature of which we do not specify now.

Let $\ell_1, \ell_2, \ldots$ be the sequence of loads which are to be applied at each oscillation so that at the ith oscillation load ℓ_i is imposed. We suppose that the loading is *cyclic* in the sense that for some $m > 1$ and all $i = 1, \ldots, m$

$$\ell_{jm+i} = \ell_{km+i} \quad \text{for all } j \neq k. \tag{3.1}$$

and

$$\ell_{i+1}(0) = \ell_i(1).$$

Hence the $(j+1)$st cycle is the loading $(\ell_{jm+1}, \ldots, \ell_{jm+m})$ and the total damage, in the sense of extension of the crack, during the cycle of loads is for $j = 0, 1, 2, \ldots$

$$Y_{j+1} = X_{jm+1} + \cdots + X_{jm+m} \tag{3.2}$$

where X_{jm+i} is the microscopic crack extension due to load ℓ_i applied in the ith oscillation of the $(j+1)$st cycle. We now set

$$S_n = Y_1 + \cdots + Y_n \tag{3.3}$$

as the total crack length at the end of n cycles.

Now we make the assumption 1° below corresponding to (a) in Section 2, namely that loads which are the same in any cycle force a crack extension which has the same distribution no matter where they occur in the sequence of loads. Specifically, we assume

1° If the load ℓ is applied at the ith oscillation, then the incremental crack extension X_i is a random variable with a probability distribution depending only upon ℓ.

This assumption implies the independence of crack extension in each oscillation from the total crack length, as well as the order of the loads. While this may be realistic in the early stages of fatigue crack growth, it is not applicable in many practical situations, and modified assumptions will be proposed in Section 4.

We also make an assumption about the behavior of this incremental crack extension. Specifically, we assume

2° For each $i=1,2,...$ the random variable X_i is non-negative and has a distribution with an increasing failure rate.

We believe that this assumption is realistic, at least during the stable phase of crack growth, by the following reasoning: a well-accepted mechanism of crack growth says that the longer the crack propagates during an oscillation the more blunt it becomes (with the consequent relaxation of the stress intensity at the crack tip) and the farther into unworked material it progresses. Assumption 2° is compatible with such an explanation since the concept of increasing failure rate applied to incremental crack lengths can be interpreted as meaning that the longer the incremental crack has grown as a result of one oscillation, the more likely it will not propagate a given distance farther. For a discussion of the definition and properties of increasing failure rate (IFR) distributions (or random variables), see [1].

Perhaps assumption 2° can be made more plausible by the following argument: Consider molecular bonds holding at the tip of the crack within

the plastic zone of the metal. The stress due to the load should attenuate away from the crack tip. It is reasonable that for a given stress the probability of the successive rupture of each molecular bond given the preceding one has ruptured must decrease since the stress is relieved. But it follows that this is equivalent with the random number of bonds broken during each oscillation being IFR. This assertion is made explicit in Section A of the appendix.

If W is the total crack length at which failure occurs, the number of such cycles until failure is the integer valued random variable N defined by the event

$$[S_{N-1} < W, \ S_N \geq W].$$

Note that failure as we have used it may mean anything from "catastrophic rupture" to "the crack is of such a length that it is inspectable". For this and other reasons one might consider the crack length W which is defined as failure to be a random variable with a given distribution function G. Now we make our last assumption:

3° The crack length W is statistically independent

of the crack length S_n for all n=1,2,... .

This means simply that knowing the length w at which failure will take place has no influence upon the behavior of the crack growth.

The conditional distribution of N given W is for n=1,2,...

$$p_n(w) = P[N=n \mid W=w] = P[S_{n-1} < w, \ S_n \geq w]$$

$$p_n(w) = H_{n-1}(w) - H_n(w)$$

(3.4)

where H_n is the distribution of S_n for n=1,2,... and $H_0 \equiv 1$. Thus the distribution of N is given by

$$P[N \leq n] = \int_0^\infty \sum_{j=1}^n p_j(w) \, dG(w) = \int_0^\infty [1 - H_n(w)] \, dG(w). \tag{3.5}$$

If we let ν denote the expected number of cycles to failure, then

$$\nu = EN = \int_0^\infty \sum_{n=1}^\infty n p_n(w) \, dG(w) = \int_0^\infty \sum_{n=0}^\infty H_n(w) \, dG(w). \tag{3.6}$$

Notice that in our case, from our assumption of periodic loading, the Y_j for $j=1,2,\ldots$ are identically distributed. We label the common distribution F. Then H_n is merely the n-fold convolution of F with itself. Now let the expected crack increment per cycle be denoted by

$$\mu = EY_j = \int_0^\infty x \, dF(x)$$

and we can utilize our notation to state a well-known result from renewal theory which is fundamental in what follows:

<u>Lemma</u>: If $Y_1, Y_2, \ldots$ are independently and identically distributed non-negative random variables, with H_n the distribution of $Y_1 + \cdots + Y_n$, then for all $w > 0$ we have the left-hand inequality holding

$$\frac{w}{\mu} - 1 \leq \sum_{n=0}^\infty H_n(w) \leq \frac{w}{\mu} \tag{3.7}$$

while if the Y_j are also IFR, then the right-hand inequality holds as well. A complete proof of this is given on page 53-54, reference [1].

If we denote the expected value of W by

$$\tau = EW = \int_0^\infty w \, dG(w)$$

and substitute the results (3.7) into equation (3.6) we obtain the fundamental inequality involving the means

$$\frac{\tau}{\mu} - 1 \leq \nu \leq \frac{\tau}{\mu} . \tag{3.8}$$

Since we can expect in our application that $\frac{\tau}{\mu}$ is quite large we could interpret (3.8) as equality for practical purposes. If we did so we would have what is indeed Miner's rule in a different form as we now show. If we let X_i have distribution $F(\cdot, \ell_i)$ and

$$\mu_i = EX_i = \int_0^\infty x dF(x : \ell_i) \qquad i=1,\ldots,m$$

be the expected length of the crack extension due to the ith load oscillation it then follows from (3.2) that

$$\mu = \sum_{i=1}^m \mu_i . \tag{3.9}$$

Of course, if we consider the derivation above in the special case where all the imposed stresses are the same for each oscillation, resulting from the same load function ℓ_i, we would have from (3.8)

$$\frac{\tau}{\mu_i} - 1 \leq \nu_i \leq \frac{\tau}{\mu_i} \tag{3.10}$$

where ν_i is the expected number of oscillations until failure under ℓ_i. Substituting (3.9), (3.10) into (3.8) and simplifying yields the following inequality

$$\frac{1}{\sum_1^m \frac{1}{\nu_i}} - 1 \leq \nu \leq \frac{1}{\sum_1^m \frac{1}{\nu_i+1}} . \tag{3.11}$$

In view of the assumption of periodicity as stated in (3.1), there are only a finite number, in fact not more than m, distinct load functions which can be assumed in any one oscillation of any cycle. Say, that the number of distinct loads is k, then by virtue of $1°$ we can assume $\ell_1, \ldots, \ell_k$ are those distinct load functions. If also during all m oscillations of the cycle the load ℓ_i occurs n_i times, $i=1, \ldots, k$, $\sum_{i=1}^{k} n_i = m$, then we have from (3.11) that

$$\frac{1}{\sum_{1}^{k} \frac{n_i}{\nu_i}} - 1 \leq \nu \leq \frac{1}{\sum_{1}^{k} \frac{n_i}{\nu_i + 1}} . \tag{3.12}$$

This formula is a direct analog of (2.5) stated in terms of mean values and in this form is precise.

We now point out that the same formula (3.12) holds in the case of randomized stresses with the proper interpretation of the notation. Suppose that during each oscillation of the cycle each of the k distinct loads $\ell_1, \ldots, \ell_k$ may occur at random with probabilities $r_1, \ldots, r_k$, respectively; then we obtain by a similar argument that (3.12) holds but n_i now denotes the expected number of oscillations of load ℓ_i during the cycle: $n_i = r_i m$.

Now we note that in both cases, programmed as well as randomized stress, we can write

$$EN \doteq \frac{1}{\sum_{i=1}^{k} \frac{n_i}{EN_i}} \tag{3.13}$$

as an approximate equality since one sees that in fact

$$\frac{1}{\sum \frac{n_i}{\nu_i}} - 1 \leq \frac{1}{\sum \frac{n_i}{\nu_i}} \leq \frac{1}{\sum \frac{n_i}{\nu_i + 1}} .$$

Thus under assumptions 1°, 2°, 3° we see (3.13) is a correct re-interpretation of Miner's rule in the form (2.5) stated in terms of expected values. Hence under any physical conditions for which these assumptions are approximately true we might expect good agreement on the average.

It is necessary, however, to delineate carefully what has been claimed in (3.13), and to point out that, if some assumptions made in deriving this formulation of Miner's rule are not satisfied, (3.13) cannot be justified and indeed, may be incorrect. Two such instances are now presented in what follows.

The usual interpretation of Miner's rule takes into account only the maximum values of the loads (in fact, n_i is often taken as the number of loads with the same peak value as ℓ_i) and we do not claim that this is correct procedure. In fact, we specifically state that it is the number of loads which are exactly equal to ℓ_i which are to be counted. Thus if one wishes to use only the maximum value of the loads one must first utilize some theory to reduce the actual imposed load to its equivalent. For example, if one believes that the expected incremental crack growth per oscillation (in some other terminology this is called the crack growth rate) for a given load ℓ is proportional to

$$(\max \ell)^a (\max \ell - \min \ell)^b, \qquad (3.14)$$

for some known $a, b > 0$, (an assumption made for example in [4] and the article by W. Weibull, page 201 in [7]), then we must reduce this to an equivalent load $\ell^°$ which has minimum stress zero. We find

$$\max \ell^\circ = (\max \ell)^{\frac{a}{a+b}}(\max \ell - \min \ell)^{\frac{a}{a+b}}$$

and it is the loads $(\ell_1^\circ, \ldots, \ell_m^\circ)$ to which one should apply Miner's rule when taking account only of maximum values of the load.

It has also been stated that all theories, for which the expected crack growth per oscillation depends only upon the load during that oscillation (such as in assumption 1°) must fail to account for such empirically demonstrated behavior as crack deceleration, crack acceleration, crack jump and crack arrest. To see that this crticism is at least partially justified we consider the following fairly general model.

We utilize the evidence given e.g. page 57 [4] that incremental crack growth takes place only during the stress rise portion of an oscillation. Suppose we take as the expected incremental growth per oscillation

$$\mu_j = EX(\ell_j) = \int_0^1 h[\ell_j(t)][\ell_j'(t)]^+ dt \tag{3.15}$$

where $x^+ = \begin{matrix} x & \text{if} & x > 0 \\ 0 & \text{if} & x < 0 \end{matrix}$ and h is some non-negative function.

It can be seen that for any h the expected crack extension during any loading, such as is drawn in Section B of the appendix would be exactly the same if one integrates only along the rise portion, in the manner of (3.15) whether the cycle was reversed or not. This claim is detailed in Section B of the appendix.

However, experimental evidence such as reported in [8] shows that reversing the cyclic order alters the mean crack extension significantly. This effect cannot be accounted for by models using assumption 1°, e.g. as exemplified by (3.15).

4.   A Probabilistic Model II

It is clear that assumption 1° of the preceding section is the
one upon which our results depend most heavily but also it is the one
that is the most open to question. In fact, careful experimentation
shows that it would seem to be false under many conditions,
(see [4], [5]). Thus in order to obtain a model which is consistent with
knowledge gained from study of fracture mechanics and metallurgy we make some
adaptations in 1° and 2° but throughout this section we assume 3° holds.

We now suppose that the loading is cyclic in the sense of (3.1)
and that

$$\ell_{jm}(0) = 0 \quad \text{for} \quad j=1,2,\dots , \tag{4.1}$$

and instead of 1° we propose the assumption

1' The incremental crack extension X_i following the
application of the ith oscillation is a random variable
with a distribution which depends upon the size of the
crack at the start of the cycle, upon the actual
incremental extensions caused by oscillations which have
preceded X_i during that cycle, and a certain number,
say r, of the preceding loads.

Note that the actual extensions during preceding cycles have no influence
upon subsequent cycles. This assumption is based on the empirical evidence
of the influence of the crack size, the prehistory of the loading and the
resulting incremental crack extensions during those oscillations
(fluctuations) of the cycles (see for example [8]).

The loading cycles which we consider all have zero load at the start of each cycle. Thus, at that time, we believe there is no memory beyond that of the actual crack size and the most recent loads. This is plausible if each load fluctuation causes strain energy which is dissipated either in the extension of the crack or the production of a plastic zone near the crack tip or both. Even when the load is relaxed, there may remain some strain hardening or other effects caused by certain loads within the recent past which can still influence the crack growth.

Except for the initial cycles, which we neglect, the exact number determined by r, we have

$$Y_{j+1} = X_1 + \cdots + X_m \qquad (4.2)$$

where each X_i in the jth cycle has a distribution with parameters $(S_j, X_1, \ldots, X_{i-1}, \lambda_i)$ where $\lambda_i = (\ell_i, \ell_{i-1}, \ldots, \ell_{i-r})$ is the vector consisting of prior r loads. The indices of λ_i are repeated with period m as indicated in (3.1). As before $S_n = Y_1 + \cdots + Y_n$ is the total crack length at the end of the nth cycle.

Actually the incremental crack extension per oscillation might well be a function of the total crack length up to that loading including the extension caused by the previous oscillation. However, our assumption in 1' is the more general since we consider the crack length only up to the beginning of the previous cycle and an arbitrary dependence upon the incremental growths since the beginning of that cycle which includes the former as a special case.

We now make the further assumption

2' For a given value of total crack length at the end of the jth cycle, say s_j, and the observed crack extensions $x_1, \ldots, x_{i-1}$ in the cycle prior to the ith oscillation, the incremental crack extension for the ith oscillation is $X_i(s_j, x_1, \ldots, x_{i-1}, \lambda_i)$ which is a non-negative IFR random variable.

This assumption that X_i is an IFR random variable has been made before and its justification is given by the argument in Section A of the appendix. We subsequently consider several specializations of this very general assumption 1' and examine their consequences.

We first assume

1" The dependence of X_i is only upon λ_i.

This means the effect of the actual prior incremental crack extensions during that cycle and the total crack size at the beginning of that cycle have no influence on the stochastic behavior of X_i and can be neglected. It follows from 1" that the incremental extensions X_i are independent IFR random variables. This leaves us with conditions somewhat more general than 2°.

By a result of Barlow, Marshall and Proschan [2] namely that the convolution of IFR random variables is IFR, we have from the periodic nature of our loading that Y_j are independent identically distributed IFR random variables and thus our lemma (3.7) from renewal theory applies and it follows that

$\nu(\lambda)$, the expected number of such cycles $\lambda = (\ell_1, \ldots, \ell_m)$ which can be repeated before failure, must satisfy

$$\frac{\tau}{\mu(\lambda)} - 1 \leq \nu(\lambda) \leq \frac{\tau}{\mu(\lambda)} \qquad (4.3)$$

where $\tau = EW$ and $\mu(\lambda) = EY_j(\lambda)$ for $j = 1, 2, \ldots$. For while it is true that

$$EY_j(\lambda) = \sum_{i=1}^{m} EX_i(\lambda_i)$$

we see that the problem of supplying adequate data, for the expected incremental growth during a load ℓ_i when that load has been preceded by $(\ell_{i-1}, \ldots, \ell_{i-r})$ becomes magnified.

Is this formula as far as we can go toward a rederivation of Miner's rule? As it stands there is no expression possible within the present model which can enable one to give the expected life, under an arbitrary periodic load from the information contained only in the conventional S-N diagram since the S-N diagram tells us nothing about the influence of load order. Moreover, unless more specific information on the nature of the dependence of r and λ_i can be obtained nothing can be done with (4.3).

In order to apply Miner's rule and use the S-N diagram to calculate the expected life the set Λ of all load sequences of m elements must contain a subset, say Λ_0, to which assumptions $1°$, $2°$ do apply. Then the remaining obstacle is to find a transformation α from Λ into Λ_0 such that

$$\text{if } \alpha(\lambda) = \lambda°, \text{ then } \mu(\lambda) = \mu(\lambda°) \qquad (4.4)$$

that is, the random incremental crack growth per cycle under λ and λ° both have the same expectation. By using this identity, since model I now applies to λ° in Λ°, we have the same bounds holding as in (3.11) and the same approximation, namely

$$\nu(\lambda) \doteq \frac{1}{\sum_1^k \frac{n_i}{\nu_i^\circ}}$$

with $\nu_i^\circ = EX(\ell_i^\circ)$ for $i=1,\ldots,k$ where k is the number of distinct loads among λ° and n_i is the multiplicity of ℓ_i° within λ°.

Of course the determination of the transformation α is the real problem and the *sine qua non* of a generalized Miner's rule. However, this discovery awaits more data of the type which is now being generated in the investigations by Schijve, Hardrath and others.

For example if the dependence upon λ_i was actually a known function of the greatest peak load in the preceding r loads and r was known to be of moderate value, then such a formula could be developed.

We now make assumption

1''' The dependence of X_i, the ith incremental extension in the jth cycle, is upon the prior crack extensions $X_{i-1},\ldots,X_{i-r}$ and upon the prior loading $\lambda_i = (\ell_i, \ell_{i-1}, \ldots, \ell_{i-r})$ and not upon the total crack length at the beginning of the cycle.

In this case we have dependence between successive X_i's and their convolution need not be IFR. For without further specification of the type of dependence, we can only conclude that the Y_j are

independent, non-negative and identically distributed. We then have

only half the fundamental inequality holding, namely

$$\frac{\tau}{\mu(\lambda)} - 1 \le \nu(\lambda).$$

Thus if the conditions are such that the dependence of Y_j on S_{j-1} can be neglected, perhaps whenever the total crack length is not large, e.g. failure being defined as the crack becoming only large enough to be inspectable, then we see Miner's rule must always be conservative. In case the transformation α can be found and $\alpha(\lambda) = \lambda^\circ$, then one would have

$$\nu(\lambda) \ge \frac{1}{\displaystyle\sum_{i=1}^{k} \frac{n_i}{\nu_i^\circ + 1}} - 1.$$

Lastly if 1' holds, the Y_j need not be independent so that in this case Miner's rule may well be unconservative. If the influence of the dependence of the Y_j is such that the enlarging of the total crack size causes the subsequent random crack increments per fluctuation to become stochastically larger (roughly the probability that they will be larger is increased) then we have only the other half of the fundamental inequality for the expected number of cycles to failure, namely

$$\nu(\lambda) \le \frac{\tau}{\mu(\lambda)}.$$

This claim is detailed in Section C of the appendix. In such cases we have Miner's rule always being unconservative in the sense discussed here.

In order to summarize our conclusions in this study we repeat our basic premise. For a given total crack length at the end of the jth cycle, say s, and observed crack extensions $x_1, \ldots, x_{i-1}$ in the cycle prior to the ith oscillation, with the r preceding load functions $\ell_{i-1}, \ldots, \ell_{i-r}$, the incremental crack extension for the ith oscillation is a random variable depending possibly upon all these parameters:

$$X_i(\ell_i, \ldots, \ell_{i-r}, x_1, \ldots, x_{i-1}, s). \tag{4.5}$$

We have agreed that this should reasonably be assumed to be an IFR random variable. Keeping this in mind we now see that if 1° is true, all the arguments of (4.5) may be omitted except ℓ_i. In this case the crack growths per cycle become independent identically distributed IFR random variables and Miner's rule for the number of cycles until failure is true in expectation and S-N data can be utilized to predict the expected life.

If 1" is true, namely, that the arguments of (4.5) may all be omitted except $\ell_i, \ldots, \ell_{i-r}$, then the random crack growths per cycle are independent and identically distributed IFR random variables. In this case a generalized Miner's rule for the number of cycles until failure is still true in expectation but information must be provided about the influence of load order for it to be correctly utilized.

If 1'" is true, namely, that the argument s may be deleted in (4.5), then the crack growths per cycle can be regarded as only independent and identically distributed random variables not necessarily IFR. In this case Miner's rule for the number of cycles until failure is always less than or

equal to the true expected number of cycles until failure, i.e.
it is conservative.

If 1' is true and the dependence upon s is such that the crack
growths per cycle are dependent and stochastically increasing with the
total crack length, then Miner's rule is unconservative and yields a
number which exceeds the true expected number of cycles until failure.

Appendix

Section A:

Consider a macroscopic crack within a material which, to fix ideas, we picture as follows:
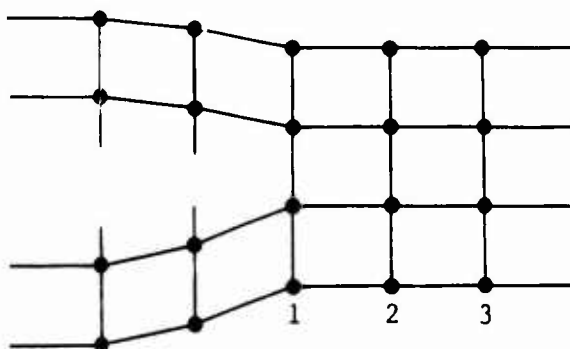
For a given stress imposed let U be the (random) number of bonds broken (unzippered). Let q_i be the probability that the ith bond is broken given that the $(i-1)^{st}$ bond is broken and hence $p_i = 1 - q_i$ is the probability the ith bond is unbroken given the $i-1^{st}$ bond is broken. Of course we assume that the probability that the ith bond is broken given that the $(i-1)^{st}$ is unbroken is zero. Hence in the multiplication theorem of conditional probabilities

$$P[U=1] = q_1 p_2$$

$$P[U=2] = q_1 q_2 p_3$$

$$\cdots \cdots \cdots$$

$$P[U=n-1] = p_n \prod_{i=1}^{n-1} q_i .$$

We now prove the

<u>Lemma</u>: The q_n are decreasing iff U is IFR.

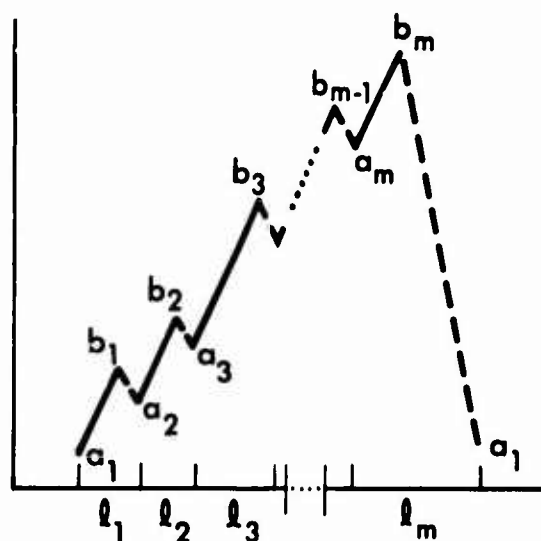<u>Proof</u>: By definition U is IFR iff $r(n)$ is increasing where

$$r(n+1) = \frac{P[U=n]}{\sum_{k=n}^{\infty} P[U=k]} = \frac{p_n \cdot \prod_{i=1}^{n-1} q_i}{p_n \prod_{1}^{n-1} q_i + p_{n+1} \prod_{1}^{n} q_i + \cdots}$$

$$= \frac{\prod_{1}^{n-1} q_i - \prod_{1}^{n} q_i}{\prod_{1}^{n-1} q_i - \prod_{1}^{n} q_i + \prod_{1}^{n} q_i - \prod_{1}^{n+1} q_0 + \cdots} = 1 - q_n,$$

from which the assertion is clear.$||$

Section B

Consider a loading cycle as follows:

The incremental extension due to the forward cycle, μ_f, is

$$\mu_f = \sum_{j=1}^{m} \mu_i = \sum_{j=1}^{m} \int_0^1 h[\ell_j(t)][\mu_j'(t)]^+ dt.$$

Integrating only along the rise portion, from a_j to b_j, for $j=1,\ldots,m$, we have

$$\mu_f = \sum_{j=1}^{m} \int_{a_j}^{b_j} h(x)dx = \sum_{j=1}^{m} [H(b_j)-H(a_j)]$$

where H is the integral of h.

Now to obtain the crack extension along the reversed cycle μ_r we integrate from a_j to b_{j-1}, $j=2,\ldots,m-1$ and a_1 to b_m

$$\mu_r = \sum_{j=0}^{m-1} \int_0^1 h[\ell_{m-j}(t)][\ell'_{m-j}(t)]^+ dt$$

$$\mu_r = H(b_m) - H(a_1) + \sum_{j=1}^{m-1} \int_{a_{m-j+1}}^{b_{m-j}} h(x)dx$$

$$= H(b_m) - H(a_1) + \sum_{j=1}^{m-1} [H(b_j) - H(a_{j+1})] = \mu_f.$$

Section C

Consider two sequences of non-negative random variables

$$Z_1, Z_2, \cdots , \qquad Y_1, Y_2, \cdots .$$

The Z_i's are independent and identically distributed by F, while the Y_i's are dependent such that Y_{n+1} depends upon $S_n = Y_1 + \cdots + Y_n$. The conditional distribution of Y_{n+1} given S_n is

$$P[Y_{n+1} \leq x | S_n = s] = G(x:s).$$

The distribution of $T_n = Z_1 + \cdots + Z_n$ is $F^{(n)}$, the n-fold convolution of F with itself. The distribution of S_n is H_n where for $n = 0, 1, 2, \ldots$

(C.1)
$$H_{n+1}(x) = \int_0^x G(x-s:s) dH_n(s)$$

with $H_0 = F^{(0)} \equiv 1$ and we assume that

(C.2)
$$G(y:0) = F(y) \quad \text{for all} \quad y > 0$$

(C.3)
$$\text{if} \quad s_1 < s_2, \quad \text{then} \quad G(x:s_1) \geq G(x:s_2).$$

In other words the distribution of the initial crack increments per cycle Z_1, Y_1 are the same but the influence of the total crack size causes the subsequent increments to become stochastically larger for the Z process.

We now prove that for all $x > 0$

$$\sum_{n=0}^{\infty} H_n(x) \leq \sum_{n=0}^{\infty} F^{(n)}(x)$$

by showing that for all $n=1,2,\ldots$

(C.4) $\qquad\qquad H_n(x) \leq F^{(n)}(x)$ for all $x > 0$.

We do this by induction. (C.4) is true for $n=1$ by (C.2). Assume it to be true for n and examine the case for $(n+1)$. Integrating (C.1) by parts we have

$$H_{n+1}(x) = \int_0^x H_n(x-y)d_y G(y:x-y) \leq \int_0^x F^{(n)}(x-y)d_y G(n:x-y),$$

by using (C.3) to show that $G(y:x-y)$ is an increasing function of y and the induction hypothesis (C.4). But reversing the integration by parts we have

$$H_{n+1}(x) \leq \int_0^x G(x-s:s)dF^n(s) \leq F^{n+1}(x)$$

since $G(x-s:s) \leq F(x-s)$ by (C.2) and (C.3). $||$

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